price of a 20 by 50 in. sheet, 10 mil in thickness, is $1.50.

5. Microscope for observing and counting tracks. This can be of very modest quality for the experimental work described here. We find the low cost Polaroid model ED-10 camera attachment to be a valuable accessory for making permanent records of the tracks observed at various stages of the etching process. The price of the ED-10 is $59.95.

VI. CONCLUSIONS

We have shown that the technique of observing charged particle tracks in solids is well within the capability of even the most modest nuclear laboratory. In addition to providing a new tool for the specialist, the technique illustrates a number of fundamental aspects of the interaction of radiation with matter and thus is of considerable pedagogic interest.

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[1] Proceedings of the International Conference on Nuclear Track Registration in Solids and Applications, Clermont-Ferrand, France, May 1969.

[2] R. L. Fleischer, P. B. Price, and R. M. Walker, Ann. Rev. Nucl. Sci. **15**, 1 (1965).

[3] R. L. Fleischer, P. B. Price, R. M. Walker, and E. L. Hubbard, Phys. Rev. **133**, A1443 (1964).

[4] G. Somogyi, Nucl. Instr. Methods **42**, 312 (1966).

Galilean Invariance and the General Covariance of Nonrelativistic Laws*

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The Galilean invariance of nonrelativistic laws is reviewed with particular attention given to the transformation properties of field-theoretic quantities. It is then shown that Galilean-invariant nonrelativistic laws generally manifest a broader covariance, the laws retaining their form under coordinate transformations to noninertial frames which move with arbitrary accelerative translational motion (without rotation) with respect to inertial frames.

1. INTRODUCTION

This paper presents a self-contained and concise review of the Galilean invariance of nonrelativistic laws, with particular attention given to the transformation properties of the dependent variables in field equations.[1] It is then shown by the study of examples that Galilean-invariant nonrelativistic laws generally manifest a broader covariance; namely, such equations generally retain their form under coordinate transformations to noninertial frames which move with arbitrary accelerative translational motion (without rotation) with respect to inertial frames. A nonrelativistic precursor to the fundamental covariance property of physical laws in Einstein's general relativity, the nonrelativistic covariance can often be exploited to facilitate the solution of nonrelativistic field equations.

2. GALILEAN INVARIANCE

The nonrelativistic physics of Galileo and Newton is applicable to physical motion involving material velocities small in absolute magnitude compared to the speed of light, and it is postulated that the basic laws of nonrelativistic physics take the same form in all inertial frames of reference. By definition, an "inertial frame" is one which moves with a constant velocity relative to the "fixed" (i.e., distant) stars. Both space and time are assumed homogeneous, and space is also assumed isotropic. Hence, inertial frames with Cartesian spatial coordinates are related by transformations of the form

$$\mathbf{x}'=\mathbf{x}+\boldsymbol{\alpha} \qquad \text{(homogeneity of space)}, \quad (2.1)$$

$$t'=t+\beta \qquad \text{(homogeneity of time)}, \quad (2.2)$$

$$x_i'=\rho_{ij}x_j \qquad [\rho_{ik}\rho_{jk}=\delta_{ij},\ \det(\rho_{ij})=+1] \quad \text{(isotropy of space)}, \quad (2.3)$$

where the components of $\boldsymbol{\alpha}$, β, and ρ_{ij} are real constant parameters. Moreover, a frame of reference which moves with constant velocity relative to an inertial frame is, by definition, an inertial frame, and thus inertial frames of reference with Cartesian spatial coordinates are related by the so-called *pure Galilean transformations*[2]:

$$\mathbf{x}'=\mathbf{x}+\boldsymbol{\gamma}t, \qquad t'=t \qquad \text{(relativity principle of Galileo and Newton)}, \quad (2.4)$$

where the components of $\boldsymbol{\gamma}$ are real constant parameters. Relating all inertial frames with Cartesian spatial coordinates, a general combination of the transformations (2.1)–(2.4) provides a generic element of the 10-parameter Galilean Lie group[3] [three independent real parameters being required for the proper rotation matrix (ρ_{ij})]. Galileo's principle of inertia (Newton's first law of motion), asserting that a point–mass particle maintains a constant velocity with respect to an inertial frame if no external forces act on the particle, is a *Galilean-invariant* physical law, in the sense that if the principle holds in an inertial frame then it will hold in any other inertial frame by logical implication.

Newton's gravitational theory for the motion of n point–mass particles is the prototype of a Galilean-invariant physical theory. In an inertial frame we have the Lagrangian for the theory prescribed as

$$L=\frac{1}{2}\sum_{i=1}^{n} m_i\,|\,\dot{\mathbf{x}}^{(i)}\,|^2+G\sum_{i<j}\frac{m_i m_j}{|\,\mathbf{x}^{(i)}-\mathbf{x}^{(j)}\,|}, \quad (2.5)$$

in which $\mathbf{x}^{(i)}=(x_1^{(i)}, x_2^{(i)}, x_3^{(i)})$ denotes the Cartesian spatial coordinates of the ith particle, m_i denotes the mass of the ith particle, and $G=6.67\times10^{-8}\,\text{cm}^3/\text{g-sec}^2$ is the universal gravitational constant. The Lagrangian (2.5) is obviously invariant under inertial frame transformations (2.1)–(2.3), with primed variables simply replacing the $\mathbf{x}^{(i)}$'s and t, while under inertial frame transformations (2.4) we find

$$L'=L+\sum_{i=1}^{n} m_i(\boldsymbol{\gamma}\cdot\dot{\mathbf{x}}^{(i)}+\tfrac{1}{2}\,|\,\boldsymbol{\gamma}\,|^2)$$

$$=L+\frac{d}{dt}\left(\sum_{i=1}^{n} m_i(\boldsymbol{\gamma}\cdot\mathbf{x}^{(i)}+\tfrac{1}{2}\,|\,\boldsymbol{\gamma}\,|^2 t)\right). \quad (2.6)$$

Because L' differs from L simply by a total time derivative, the Euler–Lagrange equations of motion derived from (2.5),

$$\ddot{\mathbf{x}}^{(i)}+G\sum_{j\neq i}[m_j(\mathbf{x}^{(i)}-\mathbf{x}^{(j)})/|\,\mathbf{x}^{(i)}-\mathbf{x}^{(j)}\,|^3]=0, \quad (2.7)$$

are invariant in form or *covariant* under Galilean transformations, Eqs. (2.7) assuming the same form (with primed variables simply replacing the $\mathbf{x}^{(i)}$'s and t) for all inertial frames of reference. In fact, Eqs. (2.7) are covariant under an 11-parameter Lie group obtained by adding to the 10-parameter transformations of the Galilean Lie group the space–time scale dilatations of the form[4]

$$\mathbf{x}'=\lambda^2\mathbf{x},$$

$$t'=\lambda^3 t, \quad (2.8)$$

where λ is a positive real constant parameter. Finally, we note that the Lagrangian (2.5) and equations of motion (2.7) are invariant under two discrete transformations, namely, space inversion with $\mathbf{x}'=-\mathbf{x}$ and time reflection with $t'=-t$.

Galilean invariance is generally featured by nonrelativistic classical field theories which do not tacitly involve a fixed frame of reference.[5] As an example, consider the fluid dynamical equations for compressible flows,

$$(\partial\mathbf{u}/\partial t)+\mathbf{u}\cdot\nabla\mathbf{u}-\nu\nabla^2\mathbf{u}+\rho^{-1}\nabla P-\mathbf{g}=0, \quad (2.9a)$$

$$(\partial\rho/\partial t)+\mathbf{u}\cdot\nabla\rho+\rho\nabla\cdot\mathbf{u}=0, \quad (2.9b)$$

$$(\partial\theta/\partial t)+\mathbf{u}\cdot\nabla\theta-D\nabla^2\theta=0, \quad (2.9c)$$

where $\mathbf{u}$ is the fluid velocity field, ρ is the density field, P is the pressure field, $\theta=\theta(\rho, P)$ is either specific entropy or temperature, $\mathbf{g}$ is the gravitational force per unit mass (possibly dependent on $\mathbf{x}$ and t), and ν, D are diffusivity constants. Equations (2.9) are obviously invariant under inertial frame transformations (2.1) and (2.2) with $\mathbf{u}'\equiv\mathbf{u}$, $\rho'\equiv\rho$, $P'\equiv P$, and $g'\equiv g$, because the space–time coordinates only appear explicitly in the differential coefficients. The Cartesian vector quality of (2.9a) and scalar quality of (2.9b) and (2.9c) guarantee that the equations are covariant under inertial frame transformations (2.3) with $u_i'\equiv\rho_{ij}u_j$, $\rho'\equiv\rho$, $P'\equiv P$, and $g_i'\equiv\rho_{ij}g_j$. In the case of pure Galilean transformations (2.4), we have the chain-rule formulas

$$\nabla'=\nabla, \qquad (\partial/\partial t')=(\partial/\partial t)-\boldsymbol{\gamma}\cdot\nabla, \quad (2.10)$$

with ∇' understood to be taken at constant t' and $\partial/\partial t'$ understood to be taken at constant $\mathbf{x}'$. Since a velocity field $\mathbf{u}$ must transform like $d\mathbf{x}/dt$ in particle mechanics, it also follows from (2.4) that

$$\mathbf{u}'=\mathbf{u}+\boldsymbol{\gamma}, \quad (2.11)$$

and hence with (2.10) we find the relation

$$(\partial/\partial t')+\mathbf{u}'\cdot\nabla'=(\partial/\partial t)+\mathbf{u}\cdot\nabla. \quad (2.12)$$

The latter transformation formula makes it evident that Eqs. (2.9) are covariant under the pure Galilean transformations (2.4) with $\rho'\equiv\rho$, $P'\equiv P$, and $\mathbf{g}'\equiv\mathbf{g}$. Thus, the dynamical equations (2.9) for compressible fluid flows assume the same form in all inertial frames of reference. Although Eqs. (2.9) are covariant under space inversion with $\mathbf{x}'=-\mathbf{x}$ and the dependent variables transforming appropriately, these dynamical equations are not covariant under time reflection nor under any prescribed space–time scale dilatations analogous to (2.8).

Galilean invariance is also featured by nonrelativistic quantum equations of motion. As an example, consider the Schrödinger equation for a spinless particle,

$$[i\hbar(\partial/\partial t)+(\hbar^2/2m)\nabla^2-V]\psi=0, \quad (2.13)$$

where ψ is the complex-valued wave function, $V=V(\mathbf{x}, t)$ is a prescribed (real scalar) potential energy associated with the particle, m is the particle mass constant, and $\hbar=1.0545\times10^{-27}$ g-cm^2/sec is the universal quantum constant. The invariance of Eq. (2.13) under inertial frame transformations (2.1)–(2.3) is obvious with $\psi'\equiv\psi$ and $V'\equiv V$. For pure Galilean transformations (2.4), we seek a transformation law of the form

$$\psi'=e^{i\phi}\psi, \quad (2.14)$$

where $\phi=\phi(\mathbf{x}, t, \boldsymbol{\gamma})$ is a real phase function such that $\phi(\mathbf{x}, t, \mathbf{0})=0$. With (2.14), the chain-rule formulas (2.10), and the physical requirement that the potential energy be the same in all inertial frames, $V'\equiv V$, we find that

$$\phi=(m/\hbar)(\boldsymbol{\gamma}\cdot\mathbf{x}+\tfrac{1}{2}\,|\,\boldsymbol{\gamma}\,|^2t) \quad (2.15)$$

as a consequence of the required covariance,

$$[i\hbar(\partial/\partial t')+(\hbar^2/2m)\nabla'^2-V']\psi'=0. \quad (2.16)$$

It is the *physical admissibility* of the transformation formula (2.14), (2.15) which guarantees that Schrödinger's equation (2.13) assumes the same form (2.16) in all inertial frames of reference.[5] To elucidate the physical admissibility of the transformation formula (2.14), (2.15), let us recast the Schrödinger equation (2.13) into an equivalent system of pseudofluid dynamical equations,[6]

$$(\partial\mathbf{u}/\partial t)+\mathbf{u}\cdot\nabla\mathbf{u}+m^{-1}\nabla V$$
$$-(\hbar^2/2m^2)\nabla(\rho^{-1/2}\nabla^2\rho^{1/2})=0,$$
$$(\partial\rho/\partial t)+\mathbf{u}\cdot\nabla\rho+\rho\nabla\cdot\mathbf{u}=0 \quad (2.17)$$

involving the real variables

$$\mathbf{u} \equiv (\hbar/2im)(\psi^{-1}\nabla\psi - \psi^{*-1}\nabla\psi^*), \quad (2.18)$$

$$\rho \equiv |\psi|^2 \equiv \psi^*\psi. \quad (2.19)$$

The transformation law (2.14), (2.15) for the wave function is such that the "electron fluid velocity" (2.18) transforms according to (2.11), while the "electron fluid density" (2.19) is the same in all inertial frames, $\rho' = \rho$. It is interesting to note that (2.13) and the equivalent system of equations (2.17) are invariant under space inversion and time reflection, with $\psi' \equiv \psi^*$ in the case of the latter, but covariance under space–time scale dilatations is not a feature of the Schrödinger equation for general V.

3. THE GENERAL COVARIANCE OF NONRELATIVISTIC LAWS

Einstein's principle of equivalence[7] removes the preferential status given to inertial frames of reference in the physics of Galileo and Newton. For a mass–point particle acted upon by a nongravitational as well as by a gravitational force field in an inertial frame, Newton's equation of motion takes the form

$$m\ddot{\mathbf{x}} = \mathbf{F} + m\mathbf{g}, \quad (3.1)$$

where m denotes the particle mass constant, $\mathbf{F}$ denotes the nongravitational force, and $\mathbf{g}$ denotes the gravitational force per unit mass. It follows from (3.1) that for a uniform gravitational field with $\mathbf{g}$ identically constant, gravitational effects disappear in a uniformly accelerating frame of reference with the Cartesian spatial coordinates $\mathbf{x}' = \mathbf{x} - \frac{1}{2}t^2\mathbf{g}$. Furthermore, Eq. (3.1) retains its form in the coordinates

$$\mathbf{x}' = \mathbf{x} + \boldsymbol{\xi}(t), \qquad t' = t, \quad (3.2)$$

of any frame of reference which moves with arbitrary accelerative translational motion (without rotation) with respect to inertial frames if the gravitational force per unit mass transforms as

$$\mathbf{g}' = \mathbf{g} + \ddot{\boldsymbol{\xi}}(t). \quad (3.3)$$

In Eqs. (3.2) and (3.3), $\boldsymbol{\xi}(t)$ denotes an arbitary twice-differentiable vector function of time. Manifested in an obvious fashion by Eq. (3.1), this covariance is featured by all Galilean-invariant nonrelativistic laws, as illustrated by example below, and represents an important natural extension of Galilean invariance with the coordinate transformation formula (3.2) superseding (2.1) and (2.4).[8] A general combination of the transformations (2.2), (2.3), and (3.2) provides a generic element of an infinite-dimensional Lie covariance group with the three components of $\boldsymbol{\xi}(t)$ arbitrary twice-differentiable functions of time.

Consider the covariance under (3.2) of the fluid dynamical Eq. (2.9) for compressible flows. From (3.2) we obtain the chain-rule formulas

$$\nabla' = \nabla, \qquad (\partial/\partial t') = (\partial/\partial t) - \dot{\boldsymbol{\xi}}(t)\cdot\nabla \quad (3.4)$$

and the transformation law for a velocity field

$$\mathbf{u}' = \mathbf{u} + \dot{\boldsymbol{\xi}}(t), \quad (3.5)$$

and thus the transformation formula (2.12) is valid for all $\boldsymbol{\xi}(t)$. Hence, with (2.12), (3.3), (3.5), $\rho' \equiv \rho$, and $P' \equiv P$, it is evident that Eqs. (2.9) are covariant under the transformations (3.2), the dynamical equations (2.9) for compressible fluid flows assuming the same form, with the gravitational force per unit mass transforming according to (3.3), in all frames which move with arbitrary accelerative translational motion with respect to inertial frames. In particular, if $\mathbf{g}$ is independent of $\mathbf{x}$, the transformed gravitational force per unit mass (3.3) vanishes for $\boldsymbol{\xi}(t)$ such that $\ddot{\boldsymbol{\xi}}(t) = -\mathbf{g}$, and hence buoyancy forces do not appear in such suitably prescribed accelerative frames of reference.[9]

Consider the covariance under (3.2) of the Schrödinger equation (2.13) for a spinless particle. With the generalized form of the transformation law (2.14), (2.15),

$$\psi' = \left[\exp\left(\frac{im}{\hbar}\right)\left(\dot{\boldsymbol{\xi}}\cdot\mathbf{x} + \frac{1}{2}\int_0^t |\dot{\boldsymbol{\xi}}|^2 dt\right)\right]\psi, \quad (3.6)$$

the chain-rule formulas (3.4), and the principle of equivalence[7] requirement

$$V' = V - m\ddot{\boldsymbol{\xi}}\cdot\mathbf{x}, \quad (3.7)$$

it follows that (2.16) is a formal consequence of (2.13) for all $\boldsymbol{\xi}(t)$. Thus, the Schrödinger Eq. (2.13) is covariant under the transformations (3.2) with the wave function and potential energy transforming according to (3.6) and (3.7). This covariance of the Schrödinger equation under coordinate transformations to noninertial frames which move with arbitrary accelerative translational motion can be utilized to solve certain time-dependent potential problems in quantum mechanics.[8]

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[1] Previous treatments of Galilean invariance and the covariance of nonrelativistic laws have been presented by A. Trautman in *Perspectives in Geometry and Relativity,* edited by B. Hoffman (Indiana U. P., Bloomington, Indiana, 1966), and works cited therein; Commun. Math. Phys. **6,** 248 (1967); O. Heckmann and E. Schücking, Z. Astrophys. **38,** 95 (1955); **40,** 81 (1956); J. Ehlers and W. Rienstra, Astrophys. J. **155,** 105 (1969).

[2] Inertial frames in nature are related *precisely* by Lorentz transformations; a Galilean transformation (2.4) approximates a Lorentz transformation for $|\gamma|\ll c=3\times10^{10}$ cm/sec, the speed of light in vacuum.

[3] A generic element of the group is expressed canonically [see, for example, G. Rosen, *Formulations of Classical and Quantum Dynamical Theory* (Academic, New York, 1969), Appendix B] as $(\mathbf{x}', t')=[\exp(\alpha\cdot G)](\mathbf{x}, t)$, where the generators associated with (2.1)–(2.4) are: $G_i\equiv\nabla_i$, $G_4\equiv\partial/\partial t$, $G_{i+4}\equiv\epsilon_{ijk}x_j\nabla_k$, $G_{i+7}\equiv t\nabla_i$. By working out the commutators of the generators, it is seen that the Lie algebra $\mathfrak{A}$ associated with the full Galilean group is the semidirect sum of the six-dimensional Lie algebra $\mathfrak{A}'$ composed of all real linear combinations of $\{G_i, G_{i+7}; i=1, 2, 3\}$ and the four-dimensional Lie algebra $\mathfrak{A}''$ composed of all real linear combinations of $\{G_4, G_{i+4}; i=1, 2, 3\}$: $\mathfrak{A}=\mathfrak{A}'\oplus\mathfrak{A}''$ with $[\mathfrak{A}', \mathfrak{A}'']=\mathfrak{A}'$.

[4] Invariance of Eq. (2.7) under the space–time scale dilatations (2.8) implies that if $\mathbf{x}^{(i)}=\mathbf{f}^{(i)}(t)$ is a solution to the equations, then so is $\mathbf{x}^{(i)\prime}=\lambda^2\mathbf{f}^{(i)}(\lambda^{-3}t)$ for all $\lambda>0$; hence, for a periodic bounded motion with $\sup_t|\mathbf{f}^{(i)}(t)|\equiv A^{(i)}$ and $\mathbf{f}^{(i)}(t+T^{(i)})\equiv\mathbf{f}^{(i)}(t)$ we find $A^{(i)\prime}=\lambda^2A^{(i)}$ and $T^{(i)\prime}=\lambda^3T^{(i)}$ for the associated one-parameter family of motions and obtain a generalized version of Kepler's third law, $(T^{(i)\prime}/T^{(i)})^2=(A^{(i)\prime}/A^{(i)})^3$, for motions related by the space–time scale dilatations.

[5] For example, the Fourier heat equation $\partial T/\partial t=D\nabla^2T$ (governing the temperature distribution in a solid *at rest*) is not Galilean invariant with the physical transformation of temperature, $T'=T$. On the other hand, a formal (unphysical) "Galilean invariance" of the heat equation is admissible mathematically if the temperature is prescribed to transform as $T'=\{\exp[-(\frac{1}{2}\gamma\cdot\mathbf{x}+\frac{1}{4}|\gamma|^2t)/D]\}T$ for pure Galilean transformations (2.4), as easily verified by making use of the chain-rule formulas (2.10).

[6] E. Madelung, Z. Physik **40,** 322 (1926).

[7] See, for example, A. Einstein, *The Meaning of Relativity* (Princeton U. P., Princeton, N. J., 1955), pp. 55–60.

[8] G. Rosen, Lett. Nuovo Cimento **2,** 61 (1971).

[9] This affords an elegant way of treating upper atmospheric or other boundary-free flows in cases for which buoyancy effects are not represented accurately by the Boussinesq approximation [employed, for example, by S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability* (Oxford U. P., New York, 1961)].