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HIV SPREAD IN THE SAN FRANCISCO COHORT: SCALING OF THE EFFECTIVE LOGISTIC RATE FOR SEROPOSITIVITY

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A simple scaling (semigroup) property is manifest in the functional form of the effective logistic rate for the increase in the HIV seropositive fraction in the San Francisco (City Clinic) cohort. With $t_i = 4.5$ years, this scaling property—$r \to \lambda^{-2} r$ under $t \to [\lambda t + (\lambda - 1)t_i]$ for all parameter values $\lambda \geqq 1$—encapsulates the effects of relevant biological and sociological changes in the key epidemiological variables during the 8-year seropositive rise period, 1978–1985 inclusive.

1. Introduction. Basic qualitative features in the epidemiological transmission of the human immunodeficiency virus (HIV)—the aetiological agent of the acquired immune deficiency syndrome (AIDS)—have been clarified by the May–Anderson mathematical model (May and Anderson, 1987; Anderson and May, 1986; Anderson *et al.*, 1986; Kolata, 1987; Whyte *et al.*, 1987), but more clinical data and refined theoretical analyses are required before the model can be used to make reliable quantitative predictions for the future incidence of HIV in susceptible population groups. So far to date, the most ample (statistically incisive) clinical data pertains to the development of HIV antigen seropositivity in the San Francisco (City Clinic) cohort of 6875 homosexual and bisexual men monitored from 1978 to 1985. The purpose of the present communicaton is to point out that a quantitatively accurate formula for the seropositive fraction in the San Francisco cohort is obtainable from the May–Anderson model if the latter is supplemented with a simple scaling (semigroup) property for the effective logistic rate of increase in the proportion of seropositive individuals—the quantity $r = r(t)$ appearing in equation (4) and defined by equation (5) in terms of May–Anderson variables. Displayed below in equation (6), this scaling property encapsulates the effects of relevant biological and sociological changes in the key epidemiological variables during the period 1978–1985 for the San Francisco cohort. Additional pertinent clinical data must be compiled in order to determine whether the scaling property (6) also applies more generally to other essentially communal HIV cohort groups during the rise-period (approximately of an 8-year duration) for prevalent seropositivity.

2. Effective Logistic Rate for Seropositivity. Representative governing equations in the homogeneous-mixing version of the May–Anderson model take the form (Anderson and May, 1986)

$$\mathrm{d}X/\mathrm{d}t = \Lambda - \beta\varepsilon XY/N - \mu X \tag{1}$$

$$\mathrm{d}N/\mathrm{d}t = \Lambda - \mu N - \alpha A \tag{2}$$

for a homosexual community of $N = N(t)$ individuals at time t, where $X = X(t)$ denotes the number of individuals who are (noninfected) susceptibles, $Y = Y(t)$ is the number of seropositive infectives, $\Lambda = \Lambda(t)$ is the immigration rate of recruitment of fresh susceptibles into the homosexual community, $A = A(t)$ is the number of people with AIDS and the parameters β, ε, μ, α denote the transmission efficiency, the mean number of sexual partners per individual per unit time, the death rate of non-AIDS people and the death rate of AIDS patients, respectively. On a time scale of the order of the mean incubation time for AIDS to appear after an individual is infected and becomes seropositive, the transmission efficiency β and the mean number of sexual partners per unit time ε decrease with t, owing to an increased use of contraception, a more guarded choice of sexual partners and perhaps also immunological resistance in the community.

It follows deductively from equations (1) and (2) that the proportion (i.e. population fraction) of seropositive individuals

$$q = (N - X)/N \tag{3}$$

satisfies the generalized Verhulst logistic equation (see, for example, Rosen, 1984, 1987a)

$$\mathrm{d}q/\mathrm{d}t = r(q - q^2) \tag{4}$$

with the effective logistic rate for seropositivity $r = r(t)$ given by

$$r = (\beta\varepsilon Y - \alpha A)/(N - X) - \Lambda/X. \tag{5}$$

To derive equation (4), one simply calculates the time derivative of (3), $\mathrm{d}q/\mathrm{d}t = -N^{-1}\,\mathrm{d}X/\mathrm{d}t + XN^{-2}\,\mathrm{d}N/\mathrm{d}t$, and makes use of equations (1), (2) and definitions (3), (5) with algebraic substitutions.

3. Scaling Property. It is noteworthy that the right side of (5) does not depend on any number of individuals in an absolute fashion, but rather is a linear combination of population *ratios*, where the population sizes X, N, Y, A are determined as functions of t by (1), (2) and two additional first-order differential equations (Anderson and May, 1986). The other time-dependent quantities β, ε and Λ in (5) decrease with increasing t on a time-scale of the order of the mean incubation time, as susceptibles and infectives react to the appearance of AIDS in their community. For the San Francisco cohort, the decrease in β, ε and Λ has been such that the functional form of $r=r(t)$ manifests the (semigroup) scaling property $r\to\lambda^{-2}r$ under $t\to[\lambda t+(\lambda-1)t_i]$, i.e.

$$r(\lambda t+(\lambda-1)t_i)\equiv\lambda^{-2}r(t)\quad\text{for all }\lambda\geqslant 1 \tag{6}$$

where λ is an arbitrary dimensionless (semigroup) parameter. It follows immediately from (6) that $r=r(t)$ takes the specific functional form

$$r(t)=r_0(1+t_i^{-1}t)^{-2} \tag{7}$$

with $r_0=r(0)$ a disposable constant. [*Proof that (7) is implied by (6)*: By putting $t=0$ in (6), one gets the relation $\lambda^2 r((\lambda-1)t_i)=r(0)\equiv r_0$, from which (7) follows by setting $\lambda=(1+t_i^{-1}t)$.] A rate-function scaling property similar to (6) has been uncovered recently in stochastic fluid mechanics (Rosen, 1987b), the mechanism of suppression being nonlinear inertial reaction in this roughly analogous fluid flow context. By substituting (7) into (4), separating variables and integrating, one obtains the theoretical formula

$$q=\left(1+(q_0^{-1}-1)\exp[-r_0 t/(1+t_i^{-1}t)]\right)^{-1} \tag{8}$$

where $q_0=q(0)$ is assumed to be prescribed.

Table I compares the theoretical values for the proportion of seropositive individuals, as given by (8), and the clinically estimated values (C.D.C., 1985) for the prevalence of antibodies to HIV antigens in the San Francisco cohort. Here, $t=0$ corresponds to the end of 1978, with $q_0=0.045$ and the other constant parameters in (8) fixed as $r_0=1.469$/year and $t_i=4.50$ years. A close overall correspondence between the theoretical values given by (8) and the clinical q values is evident in Table I for all years except 1981, when a lapse in cohort test compliance may have acted to reduce the clinical estimate for q. This overall correspondence is also shown graphically in Fig. 1; the theoretical curve $q=q(t)$ given by (8) for $0\leqq t\leqq 7.3$ years appears along with histobars for the clinical q values (C.D.C., 1985; Anderson and May, 1986) in the San Francisco cohort from 1978 to 1985. In contrast to the theoretical $q(t)$ which emerge in the May–Anderson model under the assumption that β, ε and Λ are fixed constants (independent of t), the form of the theoretical $q=q(t)$ given by (8) and shown in Fig. 1 has more curvature throughout the rise and does not top out at a maximum value.

With the accumulation of comprehensive clinical data on the proportion of seropositive individuals in other essentially communal HIV cohort groups, it will be possible to check the broader generality of the scaling property (6) and associated theoretical formula (8).

TABLE I
Comparison of Theoretical and Clinically Determined Values for the Proportion of Seropositive Individuals in the San Francisco Cohort

Year end	t/1 year	q, by (8) (%)	q, clinical (%)
1978	0	4.5	4.5
1979	1	13.6	13.8
1980	2	26.5	25.9
1981	3	39.9	31.3
1982	4	51.4	51.0
1983	5	60.4	61.0
1984	6	67.3	67.3
1985	7	72.5	73.1
1986	8	76.4	
1987	9	79.4	

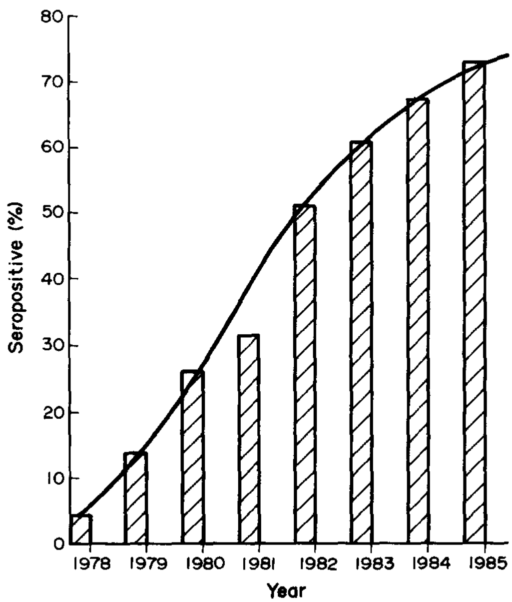

Figure 1. Histobars give the clinical q values for the proportion of seropositive individuals in the San Francisco cohort, based on the prevalance of antibodies to HIV antigens (from C.D.C., 1985; Anderson and May, 1986). Solid curve is the theoretical formula (8) for $q=q(t)$ with $q_0=0.045$, $r_0=1.469$/year, $t_i=4.50$ years and $0 \leq t \leq 7.3$ years.

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