Brief Reports

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Concise way of expressing the physicogeometric content of the gravitational field equations

Gerald Rosen
Department of Physics, Drexel University, Philadelphia, Pennsylvania 19104
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It is observed that the Einstein field equations $R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}=8\pi GT_{\mu\nu}$ are equivalent to the physicogeometric statement: A local spherical region of radius r containing energy with moment of inertia I has the physical volume $V(r)=(4\pi/3)(r^3+GI)$ in an associated free-falling frame of reference.

Simple alternative ways of expressing the fundamental laws of nature not only aid conceptual understanding but also frequently facilitate practical calculations and new theoretical extensions. Of direct experimental relevance are simple physicogeometric statements, such as the integral forms equivalent to Maxwell's differential equations in classical electromagnetic theory. The present paper reports a concise way of capsulizing the full content of Einstein's gravitational field equations

$$R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R=8\pi GT_{\mu\nu} \tag{1}$$

in terms of an equivalent physicogeometric statement which bears a manifest dependence upon the weak and strong forms of the equivalence principle in general relativity.[1]

Through gravity-free spacetime regions with the Riemannian curvature tensor $R_{\rho\mu\nu\sigma}\equiv 0$, an inertial-frame observer perceives the metric tensor components to have the Minkowski values $(\eta_{\mu\nu})\equiv\text{diag}(-1,1,1,1)$ in Cartesian rectilinear coordinates. More generally for gravitationally curved spacetime regions with $R_{\rho\mu\nu\sigma}\neq 0$, a nonrotating free-falling observer (who moves on a timelike geodesic) perceives the metric tensor components in Fermi normal coordinates as[2]

$$g_{00}\doteq -1+R_{0k0l}x_kx_l\ , \tag{2}$$

$$g_{0i}\doteq 0+\tfrac{2}{3}R_{0kil}x_kx_l\ , \tag{3}$$

$$g_{ij}\doteq\delta_{ij}+\tfrac{1}{3}R_{ikjl}x_kx_l\ . \tag{4}$$

Here the dotted equality signs mean "to second order in the Fermi normal coordinates x_1, x_2, x_3," the curvature tensor is understood to be evaluated at the observer's origin ($x_i=0$ for $i=1,2,3$) in the free-falling frame (thus the curvature tensor depends exclusively on the Fermi time coordinate x^0), and repeated latin indices are understood to be summed 1 to 3. A spatial region is by definition "local" if the metric tensor components are given accurately (to within the attainable precision of experimental measurement) by the second-order expressions (2)—(4) in a system of free-falling Fermi normal coordinates.

Consider a local rotationally symmetric spherical region in a free-falling frame of reference. For an associated system of Fermi normal coordinates with the metric tensor components given by (2)—(4), points in the spherical region have coordinates for which $|\mathbf{x}|\equiv(\sum_{i=1}^{3}x_i{}^2)^{1/2}\leq r$. The physical radius of the spherical region is precisely equal to the Euclidean coordinate value,

$$\int_0^r\sqrt{g_{11}}\,|_{x_2=x_3=0}dx_1\doteq\int_0^r(1+\tfrac{1}{6}R_{1111}x_1{}^2)dx_1=r \tag{5}$$

as a consequence of (4) and $R_{1111}\equiv 0$. On the other hand, the physical volume of the spherical region is somewhat greater than its Euclidean value $(4\pi/3)r^3$. With

$${}^3g\equiv\det(g_{ij})\doteq 1+\tfrac{1}{3}R_{ikil}x_kx_l \tag{6}$$

according to (4), the physical volume of the spherical region is

$$V(r)=\int_{|\mathbf{x}|\leq r}\sqrt{{}^3g}\,d^3x\doteq\int_{|\mathbf{x}|\leq r}(1+\tfrac{1}{6}R_{ikil}x_kx_l)d^3x =\frac{4\pi}{3}r^3+\frac{2\pi}{45}r^5R_{ikik}\ , \tag{7}$$

where use has been made of the elementary integral

$$\int_{|\mathbf{x}|\leq r}x_kx_ld^3x=\frac{4\pi}{15}r^5\delta_{kl}\ .$$

Now since the spatial components of the Ricci tensor are

$$R_{kl}=\eta^{\mu\nu}R_{\mu k\nu l}=-R_{0k0l}+R_{ikil}\ ,$$

one finds that

$$R_{ikik}=R_{kk}+R_{0k0k}=2R_{00}+R=2(R_{00}-\tfrac{1}{2}\eta_{00}R) \tag{8}$$

with introduction of the Ricci scalar $R=\eta^{\mu\nu}R_{\mu\nu}=-R_{00}+R_{kk}$ and $R_{0k0k}=\eta^{\mu\nu}R_{0\mu0\nu}=R_{00}$. Hence, by employing the "00" component of the field equations (1) in the final member of (8), substituting the latter expression into (7), and identifying the moment of inertia of energy contained in the spherical region (about an axis through its center) as $I=(8\pi r^5/15)T_{00}$, one obtains

$$V(r)=\frac{4\pi}{3}(r^3+GI) \ . \tag{9}$$

Conversely, if (9) is valid for any local spherical region of arbitrary radius r containing energy with moment of inertia I in a free-falling frame of reference, then the "00" component of the field equations (1) is required to hold in Fermi normal coordinates at $x_1=x_2=x_3=0$ for all x^0:

$$R_{00}-\tfrac{1}{2}g_{00}R=8\pi GT_{00} \ . \tag{10}$$

Since the four-velocity of the spherical region is given by $(u^0,u^1,u^2,u^3)=(1,0,0,0)$ in the free-falling frame, the scalar-invariant form of (10) is

$$(R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R-8\pi GT_{\mu\nu})u^\mu u^\nu=0 \ . \tag{11}$$

Suppose that (11), a necessary and sufficient condition for the validity of (9), is satisfied for *all* free-falling frames at a spacetime point. Then, since a Lorentz transformation based on the relative velocity of two such frames relates their respective Fermi normal coordinates at a spacetime point, the condition (11) must hold for arbitrary timelike four-velocities u^μ. It follows by a standard argument[3] that (11) implies (1), and thus the gravitational field equations (1) are satisfied if and only if (9) holds in any free-falling frame for any local spherical region of radius r containing energy with moment of inertia I.

In summary, the Einstein gravitational field equations (1) are equivalent to the physicogeometric statement: Any local spherical region of radius r containing energy with moment of inertia I in free-fall has the physical volume (9) in an associated free-falling frame of reference.

[1]For example, see R. H. Dicke, *The Theoretical Significance of Experimental Relativity* (Gordon and Breach, New York, 1964), p. 4; C. M. Will, *Theory and Experiment in Gravitational Physics* (Cambridge University Press, Cambridge, England, 1981), pp. 22–66; G. Rosen, Phys. Rev. D **31**, 1491 (1985).
[2]F. K. Manasse and C. W. Misner, J. Math. Phys. **4**, 735 (1963).
[3]See, for example, C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973), pp. 417–428.