Broadness, decay, and correlation functions of isotropic homogeneous turbulence

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(Received 30 May 1980)

General theoretical relationships consistent with experiments are obtained for isotropic homogeneous incompressible-fluid turbulence governed by the Navier-Stokes flow equation. A key quantity that structures the turbulence is the "broadness" B of the probability measure over velocity fields. Both the decay law and the longitudinal correlation function for small r appear as simple functions of this quasiconstant parameter B.

For flow of an incompressible fluid, the velocity field is solenoidal, $\vec{\nabla}\cdot\vec{u}=0$, and satisfies the Navier-Stokes equation

$$\partial\vec{u}/\partial t = -\vec{u}\cdot\vec{\nabla}\vec{u}+\nu\vec{\nabla}^2\vec{u}-\rho^{-1}\vec{\nabla}p \tag{1}$$

in which ν, ρ are positive constants. The statistical state of isotropic homogeneous turbulence is described by an ensemble of solenoidal velocity fields that evolve dynamically according to the Navier-Stokes equation (1) with a probability measure over the ensemble[1] invariant under translations and rotations of the spatial coordinates in the Galilean frame for which the mean velocity vanishes, $\langle\vec{u}(\vec{x},t)\rangle=0$. With the conventional definition[2]

$$u^2=u^2(t)\equiv\tfrac{1}{3}\langle|\vec{u}(\vec{x},t)|^2\rangle, \tag{2}$$

the rate of dissipation of the kinetic energy follows from the Navier-Stokes equation (1) as

$$\begin{aligned}\epsilon=\epsilon(t)&\equiv-\frac{d}{dt}(\tfrac{3}{2}u^2)\\ &=-\nu\langle\vec{u}\cdot\nabla^2\vec{u}\rangle=\nu\langle\nabla\vec{u}:\nabla\vec{u}\rangle,\end{aligned} \tag{3}$$

where use has been made of the homogeneous and solenoidal properties of the two-point equal-time velocity correlation tensors

$$\langle u_i(\vec{x},t)u_j(\vec{y},t)\rangle=u^2\left[\left(f+\tfrac{1}{2}r\frac{\partial f}{\partial r}\right)\delta_{ij}-\frac{1}{2r}r_ir_j\frac{\partial f}{\partial r}\right], \tag{4}$$

$$f=f(r,t),\quad \vec{r}\equiv\vec{x}-\vec{y},\quad r\equiv|\vec{r}|,$$

and $\langle u_i(\vec{x},t)u_j(\vec{x},t)u_k(\vec{y},t)\rangle$. A similar brief computation produces an expression for the time rate of change of the quantity (3), viz.,

$$\frac{d\epsilon}{dt}=2\nu\langle(\vec{u}\cdot\vec{\nabla}\vec{u})\cdot\nabla^2\vec{u}\rangle-2\nu^2\langle|\nabla^2\vec{u}|^2\rangle, \tag{5}$$

with the pressure term again vanishing because of the statistical homogeneity and $\vec{\nabla}\cdot\vec{u}\equiv0$.

To facilitate discussion of the two expectation values on the right side of (5), I introduce the "broadness" parameter

$$B\equiv\frac{\langle|\nabla^2\vec{u}|^2\rangle\langle|\vec{u}|^2\rangle}{\langle\vec{\nabla}\vec{u}:\vec{\nabla}\vec{u}\rangle^2}-1\ , \tag{6}$$

a gauge of the breadth-to-height ratio of the wave number energy spectrum,[3] or of the *broadness* of the probability measure over velocity fields itself; B is non-negative by virtue of the Schwarz inequality, with $B=0$ only for the academic δ-function energy spectrum associated with a probability measure that is concentrated entirely on Boussinesq flows of a fixed constant wave number (i.e., $\nabla^2\vec{u}=-\kappa_0^2\vec{u}\Leftrightarrow B=0$). The broadness defined by (6) is quasiconstant, changing only slowly from one period of the decay to the next, because of the statistical similarity observed[2,4-8] for all types of isotropic homogeneous turbulence in the various stages of decay. By combining the definition parts of (2) and (3) with (6), one obtains

$$\langle|\nabla^2\vec{u}|^2\rangle=(B+1)\epsilon^2/3\nu^2u^2\ , \tag{7}$$

while differentiation and contraction of (4) produces

$$\langle|\nabla^2\vec{u}|^2\rangle=u^2\nabla^4(3f+r\partial f/\partial r)|_{r=0}\,. \tag{8}$$

Hence, by equating the right-hand sides of (7) and (8), one obtains the coefficient of r^4 in alternating power series for the longitudinal correlation function f, expressible for small r as

$$f=1-\frac{r^2}{2\lambda^2}+\frac{5(B+1)}{56}\frac{r^4}{\lambda^4}-O\left(\frac{r^6}{\lambda^6}\right), \tag{9}$$

where $\lambda\equiv(15\nu u^2/\epsilon)^{1/2}$ is the Taylor microscale.[2] Since the longitudinal correlation function in (4) is directly measurable, B is deducible (along with λ) from (9) and the experimental form of f for small values of r. Thus, for example, in the case of grid-generated final period or weak turbulence, the empirical longitudinal correlation function[6]

$$f=(1+r^2/2\lambda^2)^{-1} \tag{10}$$

implies that $B=1.80$ by expansion and comparison with (9). For the case of high-Reynolds-number waterfall-generated[9] turbulence, the empirical formula[7]

TABLE I. Broadness and decay exponents for the three types of isotropic homogeneous turbulence which have been measured accurately in wind and water tunnels.

	Grid-generated initial period of decay	Grid-generated weak or final period of decay	Waterfall-generated large Reynolds number
B, broadness	5.11	1.80	37.0
n, decay exponent	1.0	2.0	3.3
Refs. for experiments	[10, 2, 4, 5, 8]	[6, 11]	[7]

$$f=1-\frac{r^2}{2\lambda^2}+3.39\frac{r^4}{\lambda^4}-O\left(\frac{r^6}{\lambda^6}\right),\quad \lambda^2=3.02\nu t\,, \tag{11}$$

and (9) yield $B=37.0$. Finally, for the initial-period of grid-generated turbulence, the typical experimental values[10] $\lambda=0.234$ in., $(d^4f/dr^4)_{r=0}=4368$ in.$^{-4}$ imply that $B=5.11$ by virtue of (9); this value for B is consistent with the lower bound obtained from correlation-function measurements in the initial period for smaller values of λ.[4] The broadness values deduced here for isotropic homogeneous turbulence are shown in Table I.

Now the experimentally established statistical similarity of isotropic homogeneous turbulence suggests that the inertial-force viscous-force scalar-product correlation in (5) may also be expressible as a universal function of B, ν, ϵ, and u^2, in a form analogous to (7) for all types of isotropic homogeneous turbulence. Since $B=0$ implies that $\nabla^2\vec{u}=-\kappa_0^2\vec{u}$, one has

$$\langle(\vec{u}\cdot\vec{\nabla}\vec{u})\cdot\nabla^2\vec{u}\rangle=-\kappa_0^2\langle(\vec{u}\cdot\vec{\nabla}\vec{u})\cdot\vec{u}\rangle=\tfrac{1}{2}\kappa_0^2\langle(\vec{\nabla}\cdot\vec{u})\,|\vec{u}|^2\rangle=0$$

for the academic but analytically important $B=0$ case. The way in which the scalar-product correlation depends on B for $B>0$ is elucidated by the Hopf Φ equation,[12] the closed-form functional integral representation[13,14] for Φ, and the derived quantity of interest

$$\langle(\vec{u}\cdot\vec{\nabla}\vec{u})\cdot\nabla^2\vec{u}\rangle=i\left(\frac{\delta}{\delta\vec{y}}\cdot\vec{\nabla}\frac{\delta}{\delta\vec{y}}\right)\cdot\left(\nabla^2\frac{\delta\Phi}{\delta\vec{y}}\right)\bigg|_{\vec{y}=0}. \tag{12}$$

A study of the functional integral on the right-side of (12) (involving invariance considerations and scaling analysis similar to the techniques used in Ref. 13, pp. 526–528) reveals a dependence on B that is slightly stronger than linear for all $B>0$. Thus, with the aid of empirical numerical factors (which may be subject to possible revision in the future), I propose the following statistical postulate: *The probability measure over velocity fields is generally such that*

$$\langle(\vec{u}\cdot\vec{\nabla}\vec{u})\cdot\nabla^2\vec{u}\rangle=B(\tfrac{2}{9}+(0.030\,015)\ln B)\epsilon^2/\nu u^2 \tag{13}$$

for any type or decay-stage of isotropic homogeneous turbulence. As shown in the following, (13) is consistent with the decay laws for the experimentally-investigated types of isotropic homogeneous turbulence.

By substituting (7) and (13) into (5), one obtains

$$\frac{d\epsilon}{dt}=-\frac{2}{3}(1+n^{-1})\frac{\epsilon^2}{u^2}\,, \tag{14}$$

where there appears the quasiconstant parameter

$$n\equiv B^{-1}[\tfrac{1}{3}-(0.090\,045)\ln B]^{-1}\,. \tag{15}$$

The definition part of (3) permits (14) to be expressed as

$$\epsilon^{-1}\frac{d\epsilon}{dt}=(1+n^{-1})u^{-2}\frac{du^2}{dt} \tag{16}$$

and integrated twice to yield

$$u^2=(\text{const})t^{-n} \tag{17}$$

for the physical final condition $u^2\to 0$, $\epsilon\to 0$ as $t\to\infty$. The decay law (17) is indeed of the observed[2,4–8,11] form, with the decay exponent n having the experimental values shown in Table I. According to the semitheoretical formula (15), n has a minimum value of 0.745 for $B=14.9$ and is greater and finite for all $B<40.5$; through the broadness interval $4.5\lesssim B\lesssim 6.5$, n is close to 1, and this may explain the general occurrence of $n=1.0$ for the initial period, notwithstanding dis-

TABLE II. Theoretical values of the decay exponent for selected values of the broadness parameter, according to Eq. (15). Values underlined have been observed experimentally and appear in Table I.

B, broadness	1.00	<u>1.80</u>	4.00	<u>5.11</u>	7.00	14.9	20.0	<u>37.0</u>	40.5
n, decay exponent	3.000	<u>1.981</u>	1.199	<u>1.050</u>	0.904	0.745	0.786	<u>3.301</u>	∞

similar grid shapes.[2,4] Table II exhibits representative n values given by (15), and the underlined entries are in close agreement with the experimental values shown in Table I. Thus, the statistical postulate expressed by (13) is wholly consistent with the experimentally observed decay laws. The academic $n=2.50$ decay for the final period[1,2] is precluded theoretically, because the associated linear-theoretic $f=[\exp(-r^2/2\lambda^2)]$ has $B=0.40$ according to (9), while (15) requires $n=6.01$ for $B=0.40$.

This work was supported by the NASA under Grant No. NSG 7491.

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[3]This is illustrated for linear-theoretic energy spectrums of the two-parameter form ($q, \kappa_0 \equiv$ positive constants): $E(\kappa) \propto \kappa^q e^{-\kappa/\kappa_0}$ with $B=2(5+2q)/(q^2+3q+2)$ and $E(\kappa) \propto \kappa^q e^{-(\kappa/\kappa_0)^2}$ with $B=2/(q+1)$; representatives from these energy spectrum families are plotted together for comparison in Ref. 1, Fig. 2(b).

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[9]The relatively large B of waterfall-generated turbulence may stem from the variation in size of the entrained air bubbles in the initiating flow region (Ref. 7), in contrast to the fixed uniform mesh size in a grid.

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