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Particlelike Solutions to Nonlinear Scalar Wave Theories

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According to a recent theorem proved by Derrick, no absolutely stable time-independent particlelike solution of finite energy is obtainable from a large class of Lorentz-covariant scalar wave theories. We study a solvable nonlinear scalar wave theory and derive a rigorous *metastable* particlelike solution of finite energy, a quasistatic solution having a rate of dissolution which is free to be arbitrarily small relative to the associated particle rest mass. Derrick's theorem notwithstanding, the specific example presented here suggests that particlelike quasistatic solutions to a nonlinear scalar wave theory may still be of some relevancy to meson field physics, where no absolutely stable but instead metastable elementary particles are present.

I. INTRODUCTION

In a recent paper, Derrick[1] proves a theorem that precludes the existence of static (time-independent) *stable* solutions of finite energy for a wide class of nonlinear wave equations, namely, for scalar (or pseudoscalar) field theories derived from a Lagrangian density of the generic form

$$\mathcal{L} = (\dot{\theta})^2 - (\nabla\theta)^2 - f(\theta), \qquad (1.1)$$

where the admissible wave field $\theta = \theta(\mathbf{x}, t)$ is a piecewise C^2 scalar function with respect to $\mathbf{x}$ and t and $f(\theta)$ is a certain piecewise C^2 function of θ. [The continuity classes prescribed here for the functions $\theta(\mathbf{x}, t)$ and $f(\theta)$ are sufficient for Derrick's original proof and for the dynamical stability generalization[2] shown in Appendix A.] According to Derrick's result, the nonlinear wave equation which follows from a Lagrangain density (1.1),

$$\ddot{\theta} - \nabla^2\theta + \tfrac{1}{2}f'(\theta) = 0, \qquad (1.2)$$

has no time-independent localized solution $\theta = \theta_0(\mathbf{x})$ that is stable[2] and with a finite static field energy (a finite "particle rest mass"),

$$E_0 = E_0[\theta_0] \equiv \int [(\nabla\theta_0)^2 + f(\theta_0)]\, d^3\mathbf{x}. \qquad (1.3)$$

Thus, within the realm of purely classical field theory, the theorem of Derrick asserts that no stable spinless particlelike solution is obtainable from a large class of Lorentz-covariant scalar wave equations.

It is important to note that Derrick's result does not preclude the existence of *metastable* particlelike solutions of finite energy, solutions having rates of dissolution which are very small relative to their quantum-theoretic characteristic frequencies $E_0/\hbar$ based on the rest mass energy (1.3). This indeed is the actual situation in nature, for there are a certain number of well-established metastable spinless elementary particles (π and K mesons), but no stable spinless elementary particle is known to exist. In the light of Derrick's theorem, the question arises as to whether metastable (quasi-static) particlelike solutions of finite energy can be derived from a nonlinear scalar field theory based on a Lagrangian density (1.1). An affirmative answer to this question is given in the present paper. By concentrating attention on a specific nonlinear scalar wave theory, we are able to present an example of a static particlelike solution which can indeed be metastable if an appropriately large value is assigned to a certain constant of integration. In other words, we obtain a rigorous quasistatic particlelike solution having a rate of dissolution which is free within the classical field theory to be arbitrarily small relative to the associated particle rest mass energy E_0.

II. A SOLVABLE NONLINEAR SCALAR WAVE THEORY

We consider the theory based on the Lagrangian density (1.1) with

$$f(\theta) \equiv -g\theta^6 \qquad (g \equiv \text{positive physical constant}), \qquad (2.1)$$

and so the associated scalar wave equation (1.2) takes the form

$$-\ddot{\theta} + \nabla^2\theta + 3g\theta^5 = 0. \qquad (2.2)$$

[1] G. H. Derrick, J. Math. Phys. 5, 1252 (1964). Also see U. Enz, Phys. Rev. **131**, 1392 (1963), and papers cited therein.

[2] That the second variation of the energy functional (1.3) about θ_0 should be nonnegative is the stability criterion evoked by Derrick. In Appendix A we show that Derrick's necessary condition for a stable θ_0, the requirement $\delta^2E_0 \geq 0$ about θ_0, is in fact necessary and sufficient for a θ_0 that is *dynamically stable* in the more general sense of Liapunov.

The singularity-free static and spherically symmetric solution to Eq. (2.2) is given by[3]

$$\theta = \theta_0 \equiv Z(Z^4 g + r^2)^{-\frac{1}{2}}, \tag{2.3}$$

in which $r \equiv |\mathbf{x}|$ and the "size parameter" Z is a free (positive or negative) real constant of integration.[4] It is an elementary matter to verify that (2.3) satisfies Eq. (2.2):

$$\begin{aligned}\nabla^2\theta_0 &= \frac{1}{r}\frac{d^2}{dr^2}(r\theta_0) = \frac{Z}{r}\frac{d^2}{dr^2}\left(\frac{Z^4 g}{r^2}+1\right)^{-\frac{1}{2}} \\ &= \frac{Z^5 g}{r}\frac{d}{dr}(Z^4 g + r^2)^{-\frac{3}{2}} = -3Z^5 g(Z^4 g + r^2)^{-5/2} \\ &\equiv -3g\theta_0^5. \end{aligned} \tag{2.4}$$

The static field energy (1.3) associated with the solution (2.3) is also computed easily:

$$\begin{aligned} E_0 &= 4\pi \int_0^\infty \left[\left(\frac{d\theta_0}{dr}\right)^2 - g\theta_0^6\right] r^2\, dr \\ &= 4\pi Z^2 \int_0^\infty \frac{(r^4 - Z^4 g r^2)}{(Z^4 g + r^2)^3}\, dr \\ &= \frac{4\pi}{g^{\frac{1}{4}}}\int_0^\infty \frac{(s^4 - s^2)\, ds}{(1+s^2)^3} = \frac{\pi^2}{2g^{\frac{1}{4}}}. \end{aligned} \tag{2.5}$$

That the static field energy $E_0 = \pi^2/2g^{\frac{1}{4}}$ is entirely independent of the size parameter Z in the solution (2.3) was to be expected, because the general expression for the static field energy (1.3) with (2.1) is a scale-invariant quantity,[4] and therefore the rest mass of any of the particlelike solutions is prefixed in the theory. Also of some interest is the nonanalytic character of E_0 about $g = 0$, showing that the nonlinear term in (2.2) is not generally amenable to a rigorous perturbation-theory treatment even if g is taken arbitrarily small.

Let us now consider the dynamical stability of the solution (2.3). With the perturbed general solution about θ_0 given by

$$\theta = \theta_0 + \sum_{k,l,m} \frac{\xi_{kl}}{r} \operatorname{Re}\left[c_{klm} e^{ikt} Y_l^m\right], \tag{2.6}$$

in which the ξ's are real functions of r, the c's are complex constants (small in magnitude but otherwise arbitrary), and the Y's are the well-known complex spherical harmonics; the linearization of (2.2) with (2.6) produces an eigenvalue equation for the ξ's

$$\frac{d^2\xi_{kl}}{dr^2} + \left[k^2 - \frac{l(l+1)}{r^2} + 15g\theta_0^4\right]\xi_{kl} = 0 \tag{2.7}$$

which must be supplemented here with the appropriate boundary conditions for a singularity-free localized perturbation,

$$\xi_{kl}(0) = 0, \qquad \lim_{r\to\infty}\left[\frac{\xi_{kl}(r)}{r}\right] = 0. \tag{2.8}$$

Equations (2.7) and (2.8) constitute a Sturm–Liouville–Schrödinger eigenvalue problem in which k^2 plays the role of an "energy" eigenvalue and the quantity

$$-15g\theta_0^4 = -15Z^4 g(Z^4 g + r^2)^{-2} \tag{2.9}$$

acts like an attractive "potential." In conformity with Derrick's theorem, there is a ground state with a negative energy eigenvalue associated with the effective potential (2.9). That is, there exists an $l = 0$ eigenfunction ξ_{k0} with k^2 a minimum and negative in value, thus with k purely imaginary, and so the associated perturbation term in (2.6) generally grows exponentially with time in a dynamically unstable fashion. By performing some straightforward analysis, the "ground state" eigenvalue min $k^2 \equiv -\lambda_0^2$ (< 0) is determined approximately in Appendix B, from which we obtain the approximate rate of exponential dissolution of the solution (2.3),

$$\lambda_0 \cong (1.9)/Z^2 g^{\frac{1}{2}}. \tag{2.10}$$

We note that λ_0^{-1} is of the order of the characteristic time for propagation of infinitesimal disturbances through the particlelike solution (2.3) ("particle radius" of the order $Z^2 g^{\frac{1}{2}}$), and so the result (2.10) is consonant with naïve physical intuition. It follows from (2.5) and (2.10) that the dissolution rate to rest energy ratio

$$\lambda_0/E_0 \cong (0.39)/Z^2 \tag{2.11}$$

can be made arbitrarily small by letting the absolute value of the size parameter $|Z|$ take on a sufficiently large value.

In summary then, the static particlelike solution (2.3) has a finite rest energy and is metastable provided that $|Z|$ is large, corresponding to a solution which is relatively small in maximum field magnitude but relatively large in spatial extension. Such a solution, one which is not highly localized or concentrated about a point in space but rather

[3] The static and spherically symmetric specialization of (2.2) produces a so-called Emden equation, familiar in astrophysics, e.g., S. Chandrasekhar, *An Introduction to the Study of Stellar Structure* (University of Chicago Press, Chicago, Illinois, 1939). However, Eq. (2.2) itself is quite distinct from the dynamical equations ordinarily encountered in astrophysics.

[4] It should be noted that the "size parameter" Z is a constant of homology, stemming from the scale invariance of Eq. (2.2), $\theta(\mathbf{x}, t) \to \mu\theta(\mu^2\mathbf{x}, \mu^2 t)$ $(\mu \neq 0)$. In general, this scale invariance gives rise to equivalence classes for the solutions to Eq. (2.2) with Z parameterizing the members of a particular equivalence class of solutions in Eq. (2.3).

global in character, is indeed more in harmony with the qualitative notion of a "classical particle" that is obtained by applying a correspondence principle argument to the quantum field theoretic description of a one-particle state. If the term "particlelike" is understood to embrace time-independent solutions of finite energy that have a rather global (instead of a highly localized) character, the specific model theory considered here suggests that such "particlelike" solutions to nonlinear scalar wave theories may still be of some relevancy in meson field physics.

Note added in proof: A paper by R. H. Hobart [Proc. Phys. Soc. (London) **82**, 201 (1963)] has come to the author's attention. Although not cited by Derrick,[1] this paper by Hobart establishes the instability of nonsingular time-independent *spherically symmetric* solutions to equations having the generic form (1.2). Derrick's theorem follows as a natural extension of Hobart's result for spherically symmetric solutions.

APPENDIX A: EQUIVALENCE OF STABILITY CRITERIA FOR GENERAL STATIC SOLUTIONS OF FINITE ENERGY

Here we show that Derrick's necessary condition for a stable θ_0, the static energy requirement

$$\begin{aligned}\delta^2 E &\equiv \tfrac{1}{2}[(d^2/d\epsilon^2)E_0[\theta_0+\epsilon\omega]]_{\epsilon=0}\\ &= \int [(\nabla\omega)^2+\tfrac{1}{2}f''(\theta_0)\omega^2]\, d^3\mathbf{x} \geq 0 \qquad \text{(A1)}\end{aligned}$$

with both $\theta_0 = \theta_0(\mathbf{x})$ and $\omega = \omega(\mathbf{x})$ independent of time and piecewise C^2 functions with respect to $\mathbf{x}$, is in fact a necessary and sufficient condition for a θ_0 that is *dynamically stable* in the sense of Liapunov.[5] To derive the dynamical stability criterion, we make the perturbed field depend on time by putting

$$\theta(\mathbf{x}, t) = \theta_0(\mathbf{x}) + \omega(\mathbf{x}) \cos kt, \qquad \text{(A2)}$$

where the constant k may be either purely real or purely imaginary and $|\omega(\mathbf{x})| \ll |\theta_0(\mathbf{x})|$ for all values of $\mathbf{x}$. By substituting (A2) into (1.2) and retaining only the terms linear in ω, we obtain an eigenvalue equation for k^2 and ω,

$$(\nabla^2 - \tfrac{1}{2}f''(\theta_0) + k^2)\omega = 0, \qquad \text{(A3)}$$

which can be recast in the form of a variational principle,

$$\begin{gathered}\delta k^2 = 0\,, \\ k^2 \equiv \int [(\nabla\omega)^2 + \tfrac{1}{2}f''(\theta_0)\omega^2]\, d^3\mathbf{x}\left[\int \omega^2\, d^3\mathbf{x}\right]^{-1}.\end{gathered} \qquad \text{(A4)}$$

Now if $f''(\theta_0)$ is piecewise continuous and if k^2 is negative for a certain admissible (piecewise C^2 function) ω, k^2 will be stationary about some negative value of k^2; we shall then have a purely imaginary eigenvalue for k and thus a dynamically unstable static solution $\theta_0(\mathbf{x})$, according to (A2). Hence, with the hypothesis that $f(\theta)$ is piecewise C^2, a comparison of (A1) and (A4) shows that Derrick's static energy requirement $\delta^2 E_0 \geq 0$ for all admissible ω is actually necessary and sufficient for a dynamically stable solution θ_0.

APPENDIX B: GROUND STATE EIGENVALUE FOR EQS. (2.7) AND (2.8)

By substituting (2.9) into Eq. (2.7) and setting

$$\rho \equiv r/Z^2 g^{\frac{1}{2}}, \quad \gamma \equiv Z^2 g^{\frac{1}{2}}\lambda_0 \equiv Z^2 g^{\frac{1}{2}}(-\min k^2)^{\frac{1}{2}}, \qquad \text{(B1)}$$

we obtain the dimensionless eigenvalue equation for the "ground state"

$$\frac{d^2\xi_\gamma}{d\rho^2} + \left[\frac{15}{(1+\rho^2)^2} - \gamma^2\right]\xi_\gamma = 0. \qquad \text{(B2)}$$

Here, the "ground state" eigenfunction $\xi_\gamma \equiv \xi_{0k}$ with $l = 0$ and $k^2 \equiv -\lambda_0^2$ a (negative valued) minimum is associated with the rate-controlling unbounded perturbation term in (2.6). That Eq. (B2) has a bound "ground state" solution with a real $\gamma > 0$ such that

$$\xi_\gamma(0) = 0, \qquad \lim_{\rho\to\infty}\left[\frac{e^{\gamma\rho}\xi_\gamma(\rho)}{\rho}\right] = 0 \qquad \text{(B3)}$$

is confirmed most readily by considering the eigenfunctions associated with a simple (mathematically tractable) potential (e.g., a square-well attractive potential) $\tilde{V}(\rho)$ such that $0 \geq \tilde{V}(\rho) \geq -15(1+\rho^2)^{-2}$ for all real positive values of ρ. Alternatively, by considering the $\gamma = 0$ eigenfunction associated with Eqs. (B2) and (B3), an eigenfunction given explicitly in closed form by the algebraic expression[6]

$$\xi_0 = (\rho - \rho^3)(1+\rho^2)^{-\frac{3}{2}}, \qquad \text{(B4)}$$

which exhibits a node at $\rho = 1$, we infer[7] the existence of one (unique) lower energy state, necessarily with $\gamma > 0$ and *without* a node occurring for some interior value of ρ. Since the "ground state" eigenfunction ξ_γ and its eigenvalue γ cannot be obtained by exact mathematical analysis, we work out two mutually corroborating approximate solutions of the eigenvalue problem in the following paragraphs. The first approximate solution is based on a novel

[5] See, for example: W. Hahn, *Theory and Application of Liapunov's Direct Method* (Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1963), pp. 5–10.

[6] E. Kamke, *Differentialgleichungen Lösungsmethoden und Lösungen* (Akademische Verlags., Leipzig, 1956), p. 494.

[7] For example: R. Courant and D. Hilbert, *Methods of Mathematical Physics* (Interscience Publishers, Inc., New York, 1953), Vol. I, p. 458.

heuristic method, while the second approximate solution involves a more direct and foolproof Rayleigh–Ritz procedure.

First, by putting

$$\xi_\gamma \equiv e^{-\gamma\rho}(1+\rho^2)^{-\frac{3}{2}}\zeta_\gamma \tag{B5}$$

into Eq. (B2), we find an equation for the new dependent variable ζ_γ,

$$(1+\rho^2)\frac{d^2\zeta_\gamma}{d\rho^2} - [2\gamma(1+\rho^2)+6\rho]\frac{d\zeta_\gamma}{d\rho} + (12+6\gamma\rho)\zeta_\gamma = 0, \tag{B6}$$

while the boundary conditions (B3) take the form

$$\zeta_\gamma(0) = 0, \qquad \lim_{\rho\to\infty}\left[\frac{\zeta_\gamma(\rho)}{\rho^4}\right] = 0. \tag{B7}$$

Note that the algebraic denominator in Eq. (B5), suggested by the exact $\gamma=0$ solution (B4), eliminates the second-order character of the $\rho = \pm i$ poles manifest in (B2) with only simple zeros evident in the coefficients of the transformed Eq. (B6). The general Fuchsian theory guarantees that the relevant solution of (B6) is analytic about $\rho = 0$ and is thus expressible as a convergent power series for $|\rho| < 1$,

$$\zeta_\gamma = \sum_{n=1}^{\infty} a_n\rho^n, \tag{B8}$$

in which the a's are given by a recurrence relation derived from (B6),

$$a_{n+2} = \frac{2\gamma}{(n+2)}a_{n+1} - \frac{(n-4)(n-3)}{(n+2)(n+1)}a_n + \frac{2\gamma(n-4)}{(n+2)(n+1)}a_{n-1} \quad (n \geq 2), \tag{B9}$$

with

$$a_1 \equiv 1, \qquad a_2 = \gamma, \qquad a_3 = \tfrac{2}{3}\gamma^2 - 1. \tag{B10}$$

From (B9) and (B10) it follows that

$$\begin{aligned} a_4 &= \tfrac{1}{3}\gamma^3 - \gamma, \qquad a_5 = \tfrac{2}{15}\gamma^4 - \tfrac{1}{2}\gamma^2, \\ a_6 &= \tfrac{2}{45}\gamma^5 - \tfrac{1}{6}\gamma^3, \ a_7 = \tfrac{4}{315}\gamma^6 - \tfrac{4}{105}\gamma^4 - \tfrac{1}{42}\gamma^2. \end{aligned} \tag{B11}$$

Now observe that the eigenvalue condition

$$\gamma = \tfrac{1}{2}(15)^{\frac{1}{2}} \cong 1.93 \tag{B12}$$

makes $a_5 = a_6 = 0$, and so (B8) reduces to the form

$$\zeta_{(15)^{1/2}} = \rho + \tfrac{1}{2}(15)^{\frac{1}{2}}\rho^2 + \tfrac{3}{2}\rho^3 + \tfrac{1}{8}(15)^{\frac{1}{2}}\rho^4 + \tfrac{15}{336}\rho^7 + O(\rho^8), \tag{B13}$$

with the terms up to order ρ^6 being put in closest possible accord with the functional form for $\rho \sim 1$ suggested by the second boundary condition in (B7). We regard the first four terms in (B13) as an asymptotic expansion for $\zeta_{(15)^{1/2}}$ with ρ of the order or not much greater than unity, an approximate form for ζ_γ in closest possible agreement with the asymptotic behavior required by the second boundary condition in (B7). Thus, we can tentatively regard (B12) and (B13) as an approximate solution to the "ground state" eigenvalue problem.

To corroborate the preceding analysis, let us set up a variational principle for the solution to Eq. (B2) and then apply a Rayleigh–Ritz approximation procedure. Here it is convenient to introduce the new independent and dependent variables

$$\begin{aligned} \phi &\equiv \tan^{-1}\rho \qquad (0 \leq \phi \leq \tfrac{1}{2}\pi), \\ \omega_\gamma(\phi) &\equiv (\cos\phi)\xi_\gamma(\rho). \end{aligned} \tag{B14}$$

In terms of these new quantities, (B2) is transformed to the equation

$$\frac{d^2\omega_\gamma}{d\phi^2} + \left[16 - \frac{\gamma^2}{(\cos\phi)^4}\right]\omega_\gamma = 0, \tag{B15}$$

which leads to the variational principle

$$\delta\gamma^2 = 0 \quad \gamma^2 \equiv \int_0^{\frac{1}{2}\pi}\left[16\omega_\gamma^2 - \left(\frac{d\omega_\gamma}{d\phi}\right)^2\right]d\phi \tag{B16}$$

with ω_γ subject to the normalization condition

$$\int_0^{\frac{1}{2}\pi}\frac{(\omega_\gamma)^2}{(\cos\phi)^4}\,d\phi = 1 \tag{B17}$$

and the boundary conditions

$$\omega_\gamma(0) = 0, \qquad \omega_\gamma(\tfrac{1}{2}\pi) = 0. \tag{B18}$$

We seek an approximate solution of the form

$$\omega_\gamma = 2\pi^{-\frac{1}{2}}(\cos\phi)^2(\alpha\sin 2\phi + \beta\sin 4\phi), \tag{B19}$$

where α and β are variational parameters, constrained by (B17) to satisfy

$$\alpha^2 + \beta^2 = 1. \tag{B20}$$

By putting (B19) into the definition part of (B16) we have

$$\gamma^2 = 3\alpha^2 + 3\alpha\beta - \tfrac{1}{2}\beta^2, \tag{B21}$$

and thus obtain the maximizing conditions for γ^2

$$\begin{aligned} (6 - 2\gamma^2)\alpha + 3\beta &= 0, \\ 3\alpha - (1 + 2\gamma^2)\beta &= 0, \end{aligned} \tag{B22}$$

which produce

$$\gamma = \tfrac{1}{2}[5 + (85)^{\frac{1}{2}}]^{\frac{1}{2}} \cong 1.88, \tag{B23}$$

as well as the mixing ratio $\alpha/\beta = \frac{1}{6}[7 + (85)^{\frac{1}{2}}] \cong 2.70$.

A comparison of (B12) and (B23) shows that the two approximate solutions of the eigenvalue problem are mutually consistent and give $\gamma \cong 1.90$ to better than 2%. Inverting the definition of λ_0 in (B1), we finally obtain the estimate stated in (2.10).