Global definition and physical interpretation of the cosmological constant

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Abstract. – Defined as a global physical entity, the cosmological constant Λ appears here as a stationary functional of the metric, the matter (dark as well as visible) and the radiation fields. Subject to compact–support variations of the fields, $\delta\Lambda = 0$ gives the metric, matter and radiation field equations. With this rigorous physical formulation, the empirical relation $\Lambda \cong 2.7\kappa\rho_o$, where ρ_o is the average energy-density of matter and radiation, follows from the spacetime average of $\kappa(L - g^{\mu\nu}\partial L/\partial g^{\mu\nu})$ through the observable four-volume of the universe on the homogeneity scale (~100 Mpc), where L is the Lagrangian of the matter and radiation fields. Hence, the notion of negative-pressure *dark energy* is obviated in favor of an energy-density relationship for the cosmological constant that derives from the physical principle $\delta\Lambda = 0$. Moreover, this formulation can be employed practically to rule out certain common-suspect free-fields as the dominant component of dark matter. In particular, it is readily shown that a massive spin-zero scalar free-field or a massive spin-one vector free-field are precluded as the dominant component of dark matter.

The conventional action principle for relativistic cosmology is [1, 2, 3]

$$\delta\int_V (\kappa^{-1}(\tfrac{1}{2}\mathrm{R} - \Lambda) + L)\sqrt{-g}\,d^4x = 0 \tag{1}$$

Under $g^{\mu\nu} \to g^{\mu\nu} + \delta g^{\mu\nu}$ with $\delta g^{\mu\nu}$ of compact-support in the interior of four-volume V and zero on the three-dimensional boundary of V, (1) generates the field equations for the metric tensor

$$R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu} \tag{2}$$

in which the stress-energy tensor appears as

$$T_{\mu\nu} \equiv L g_{\mu\nu} - 2\frac{\partial L}{\partial g^{\mu\nu}} \tag{3}$$

after Palatinì removal (to vanishing surface integrals) of the $\delta(\Gamma^{\rho}_{\mu\nu})$ terms in δL from fields with spin ½. The field equations for the matter and radiation fields also follow from (1) by way of $\int\sqrt{-g}\,(\delta L)\,d^4x = 0$ under variations of the latter fields. In the conventional action principle (1) and in the metric tensor field equations (2), the cosmological constant Λ must be prescribed external to the fields, as an *ad hoc* and enigmatic negative-pressure *dark energy*.

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There is, however, an alternative to the variational principle (1) and the *dark energy* interpretation of the cosmological constant. Instead, let us define the cosmological constant as the global physical entity

$$\Lambda \equiv \int_{V_0} (½ R + \kappa L)\sqrt{-g}\, d^4x \Big/ \int_{V_0} \sqrt{-g}\, d^4x \equiv \langle (½\mathrm{R} + \kappa L)\rangle_0 \tag{4}$$

where the four-volume V_0 is definitive, namely, the past light-cone extending out to embrace the homogeneity scale: $V_0 \equiv \{ t, \vec{x} : |\vec{x}| \le (t_0 - t) \gtrsim 100$ Mpc$\}$. Subject to compact-support variations of the metric, matter and radiation fields through the interior of V_0, $\delta\Lambda = 0$ holds if and only if the metric, matter and radiation field equations hold. Hence, the cosmological constant defined physically by (4) is a functional of the gravitational, matter and radiation fields, without any additional energy source. Moreover, Λ defined by (4) is unchanged in value under first-order variations of the fields, the value of Λ being physically stable. However, with the universal expansion and increases in the cosmic time t, Λ changes in value concomitantly with changes in the fields through the four-volume of observation V_0. Λ is entirely a surrogate of the physical fields through V_0, with the property $\delta\Lambda = 0$ underscoring its significance.

Now note that by contracting (2) with $g^{\mu\nu}$ and using (3), one obtains

$$½ R = 2\Lambda + \kappa\left(g^{\mu\nu} \frac{\partial L}{\partial g^{\mu\nu}} - 2L \right) \tag{5}$$

and thus a solution to the field equations (2) reduces (4) to its semi-evaluated form

$$\Lambda = \kappa \left\langle \left(L - g^{\mu\nu} \frac{\partial L}{\partial g^{\mu\nu}} \right) \right\rangle_0 . \tag{6}$$

In view of (6), the magnitude of the cosmological constant is related to the form of L for the matter and radiation fields, like the average energy-density of matter and radiation obtained from (3):

$$\rho_0 = \langle \mathrm{T}_{00} \rangle_0 = \left\langle L g_{00} - 2\frac{\partial L}{\partial g^{00}} \right\rangle_0 . \tag{7}$$

Indeed, the observed relation for the empirical magnitude of the cosmological constant

$$\Lambda \cong 2.7\kappa\rho_0 \left(\cong 1.25x10^{-56}\, cm^{-2}\right) \tag{8}$$

implies a constraint on L (in particular, on its dark matter terms) in order for (8) to follow from (6) and (7). With the metric approximately flat, we have $g_{\mu\nu} \cong \mathrm{diag}(-1,\ 1,\ 1,\ 1)$. Hence, the combination of (6), (7) and (8) produces a constraint on L which must be satisfied by the proper cosmological solution to the field equations:

$$\sum_{n=1}^{3} \left\langle \frac{\partial L}{\partial g^{ii}} \right\rangle_0 \cong 3.7 \langle L \rangle_0 + 6.4 \left\langle \frac{\partial L}{\partial g^{00}} \right\rangle_0 . \tag{9}$$

The latter constraint of L can be employed to rule out certain common-suspect free-fields as the predominant component of matter and radiation, i.e., the major component of dark matter. In particular, a massive spin-zero scalar free-field is precluded, for the Lagrangian

$$L = -\tfrac{1}{2} g^{\mu\nu} \phi,_{\mu} \phi,_{\nu} - \tfrac{1}{2} m^2 \phi^2 \tag{10}$$

put into (9) and with the metric tensor $g_{\mu\nu} \cong \mathrm{diag}(-1, 1, 1, 1)$ yields

$$\left\langle (\partial\phi/\partial t)^2 + |\nabla\phi|^2 + 1.37 m^2 \phi^2 \right\rangle_0 \cong 0 \tag{11}$$

which implies that $\phi \equiv 0$. Likewise, a massive spin-one vector free-field, with the Lagrangian

$$L = -\tfrac{1}{4} g^{\mu\rho} g^{\nu\sigma} f_{\mu\nu} f_{\rho o} - \tfrac{1}{2} m^2 g^{\mu\nu} \Phi_{\mu} \Phi_{\nu} \tag{12}$$

where $f_{\mu\nu} \equiv \Phi_{\mu,\nu} - \Phi_{\nu,\mu}$, is also precluded as the major component of dark matter, for (9) with (12) yields a positive-definite expression similar to (11), which implies that $\Phi_{\mu} \equiv 0$. Finally, for a phenomenological perfect fluid representation of matter and radiation [see, *e.g*., Ref. 1, pp. 186-187] (6) yields $\Lambda = \tfrac{1}{2}\kappa(\rho_0 + 3p_0)$, where ρ_0 and p_0 are the proper mass density and proper pressure averaged through V_0; hence, a phenomenological representation would require a remarkably large effective value for p_0. While enigmatic *dark energy* is obviated by the global definition for Λ shown in (4), dark matter is revealed to be seemingly more exotic.

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